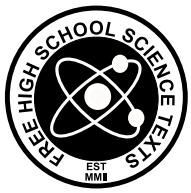

FHSST Authors

The Free High School Science Texts: Textbooks for High School Students Studying the Sciences Physics Grades 10 - 12

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Contents

I Introduction **1**

1 What is Physics? **3**

II Grade 10 - Physics **5**

2 Units **9**

2.1 Introduction . . . 9
2.2 Unit Systems . . . 9
2.2.1 SI Units . . . 9
2.2.2 The Other Systems of Units . . . 10
2.3 Writing Units as Words or Symbols . . . 10
2.4 Combinations of SI Base Units . . . 12
2.5 Rounding, Scientific Notation and Significant Figures . . . 12
2.5.1 Rounding Off . . . 12
2.5.2 Error Margins . . . 13
2.5.3 Scientific Notation . . . 13
2.5.4 Significant Figures . . . 15
2.6 Prefixes of Base Units . . . 15
2.7 The Importance of Units . . . 17
2.8 How to Change Units . . . 17
2.8.1 Two other useful conversions . . . 19
2.9 A sanity test . . . 19
2.10 Summary . . . 19
2.11 End of Chapter Exercises . . . 21

3 Motion in One Dimension - Grade 10 **23**

3.1 Introduction . . . 23
3.2 Reference Point, Frame of Reference and Position . . . 23
3.2.1 Frames of Reference . . . 23
3.2.2 Position . . . 25
3.3 Displacement and Distance . . . 28
3.3.1 Interpreting Direction . . . 29
3.3.2 Differences between Distance and Displacement . . . 29
3.4 Speed, Average Velocity and Instantaneous Velocity . . . 31

3.4.1 Differences between Speed and Velocity . . . 35
3.5 Acceleration . . . 38
3.6 Description of Motion . . . 39
3.6.1 Stationary Object . . . 40
3.6.2 Motion at Constant Velocity . . . 41
3.6.3 Motion at Constant Acceleration . . . 46
3.7 Summary of Graphs . . . 48
3.8 Worked Examples . . . 49
3.9 Equations of Motion . . . 54
3.9.1 Finding the Equations of Motion . . . 54
3.10 Applications in the Real-World . . . 59
3.11 Summary . . . 61
3.12 End of Chapter Exercises: Motion in One Dimension . . . 62

4 Gravity and Mechanical Energy - Grade 10 67
4.1 Weight . . . 67
4.1.1 Differences between Mass and Weight . . . 68
4.2 Acceleration due to Gravity . . . 69
4.2.1 Gravitational Fields . . . 69
4.2.2 Free fall . . . 69
4.3 Potential Energy . . . 73
4.4 Kinetic Energy . . . 75
4.4.1 Checking units . . . 77
4.5 Mechanical Energy . . . 78
4.5.1 Conservation of Mechanical Energy . . . 78
4.5.2 Using the Law of Conservation of Energy . . . 79
4.6 Energy graphs . . . 82
4.7 Summary . . . 83
4.8 End of Chapter Exercises: Gravity and Mechanical Energy . . . 84

5 Transverse Pulses - Grade 10 87
5.1 Introduction . . . 87
5.2 What is a *medium*? . . . 87
5.3 What is a *pulse*? . . . 87
5.3.1 Pulse Length and Amplitude . . . 88
5.3.2 Pulse Speed . . . 89
5.4 Graphs of Position and Velocity . . . 90
5.4.1 Motion of a Particle of the Medium . . . 90
5.4.2 Motion of the Pulse . . . 92
5.5 Transmission and Reflection of a Pulse at a Boundary . . . 96
5.6 Reflection of a Pulse from Fixed and Free Ends . . . 97
5.6.1 Reflection of a Pulse from a Fixed End . . . 97

5.6.2 Reflection of a Pulse from a Free End . . . 98
5.7 Superposition of Pulses . . . 99
5.8 Exercises - Transverse Pulses . . . 102

6 Transverse Waves - Grade 10 105
6.1 Introduction . . . 105
6.2 What is a *transverse wave*? . . . 105
6.2.1 Peaks and Troughs . . . 106
6.2.2 Amplitude and Wavelength . . . 107
6.2.3 Points in Phase . . . 109
6.2.4 Period and Frequency . . . 110
6.2.5 Speed of a Transverse Wave . . . 111
6.3 Graphs of Particle Motion . . . 115
6.4 Standing Waves and Boundary Conditions . . . 118
6.4.1 Reflection of a Transverse Wave from a Fixed End . . . 118
6.4.2 Reflection of a Transverse Wave from a Free End . . . 118
6.4.3 Standing Waves . . . 118
6.4.4 Nodes and anti-nodes . . . 122
6.4.5 Wavelengths of Standing Waves with Fixed and Free Ends . . . 122
6.4.6 Superposition and Interference . . . 125
6.5 Summary . . . 127
6.6 Exercises . . . 127

7 Geometrical Optics - Grade 10 129
7.1 Introduction . . . 129
7.2 Light Rays . . . 129
7.2.1 Shadows . . . 132
7.2.2 Ray Diagrams . . . 132
7.3 Reflection . . . 132
7.3.1 Terminology . . . 133
7.3.2 Law of Reflection . . . 133
7.3.3 Types of Reflection . . . 135
7.4 Refraction . . . 137
7.4.1 Refractive Index . . . 139
7.4.2 Snell's Law . . . 139
7.4.3 Apparent Depth . . . 143
7.5 Mirrors . . . 146
7.5.1 Image Formation . . . 146
7.5.2 Plane Mirrors . . . 147
7.5.3 Ray Diagrams . . . 148
7.5.4 Spherical Mirrors . . . 150
7.5.5 Concave Mirrors . . . 150

7.5.6 Convex Mirrors . . . 153
7.5.7 Summary of Properties of Mirrors . . . 154
7.5.8 Magnification . . . 154
7.6 Total Internal Reflection and Fibre Optics . . . 156
7.6.1 Total Internal Reflection . . . 156
7.6.2 Fibre Optics . . . 161
7.7 Summary . . . 163
7.8 Exercises . . . 164

8 Magnetism - Grade 10 **167**
8.1 Introduction . . . 167
8.2 Magnetic fields . . . 167
8.3 Permanent magnets . . . 169
8.3.1 The poles of permanent magnets . . . 169
8.3.2 Magnetic attraction and repulsion . . . 169
8.3.3 Representing magnetic fields . . . 170
8.4 The compass and the earth's magnetic field . . . 173
8.4.1 The earth's magnetic field . . . 175
8.5 Summary . . . 175
8.6 End of chapter exercises . . . 176

9 Electrostatics - Grade 10 **177**
9.1 Introduction . . . 177
9.2 Two kinds of charge . . . 177
9.3 Unit of charge . . . 177
9.4 Conservation of charge . . . 177
9.5 Force between Charges . . . 178
9.6 Conductors and insulators . . . 181
9.6.1 The electroscope . . . 182
9.7 Attraction between charged and uncharged objects . . . 183
9.7.1 Polarisation of Insulators . . . 183
9.8 Summary . . . 184
9.9 End of chapter exercise . . . 184

10 Electric Circuits - Grade 10 **187**
10.1 Electric Circuits . . . 187
10.1.1 Closed circuits . . . 187
10.1.2 Representing electric circuits . . . 188
10.2 Potential Difference . . . 192
10.2.1 Potential Difference . . . 192
10.2.2 Potential Difference and Parallel Resistors . . . 193
10.2.3 Potential Difference and Series Resistors . . . 194
10.2.4 Ohm's Law . . . 194

10.2.5 EMF . . . 195
10.3 Current . . . 198
10.3.1 Flow of Charge . . . 198
10.3.2 Current . . . 198
10.3.3 Series Circuits . . . 199
10.3.4 Parallel Circuits . . . 200
10.4 Resistance . . . 202
10.4.1 What causes resistance? . . . 202
10.4.2 Resistors in electric circuits . . . 202
10.5 Instruments to Measure voltage, current and resistance . . . 204
10.5.1 Voltmeter . . . 204
10.5.2 Ammeter . . . 204
10.5.3 Ohmmeter . . . 204
10.5.4 Meters Impact on Circuit . . . 205
10.6 Exercises - Electric circuits . . . 205

III Grade 11 - Physics 209

11 Vectors 211
11.1 Introduction . . . 211
11.2 Scalars and Vectors . . . 211
11.3 Notation . . . 211
11.3.1 Mathematical Representation . . . 212
11.3.2 Graphical Representation . . . 212
11.4 Directions . . . 212
11.4.1 Relative Directions . . . 212
11.4.2 Compass Directions . . . 213
11.4.3 Bearing . . . 213
11.5 Drawing Vectors . . . 214
11.6 Mathematical Properties of Vectors . . . 215
11.6.1 Adding Vectors . . . 215
11.6.2 Subtracting Vectors . . . 217
11.6.3 Scalar Multiplication . . . 218
11.7 Techniques of Vector Addition . . . 218
11.7.1 Graphical Techniques . . . 218
11.7.2 Algebraic Addition and Subtraction of Vectors . . . 223
11.8 Components of Vectors . . . 228
11.8.1 Vector addition using components . . . 231
11.8.2 Summary . . . 235
11.8.3 End of chapter exercises: Vectors . . . 236
11.8.4 End of chapter exercises: Vectors - Long questions . . . 237

12 Force, Momentum and Impulse - Grade 11 **239**

12.1 Introduction . . . 239
12.2 Force . . . 239
12.2.1 What is a *force*? . . . 239
12.2.2 Examples of Forces in Physics . . . 240
12.2.3 Systems and External Forces . . . 241
12.2.4 Force Diagrams . . . 242
12.2.5 Free Body Diagrams . . . 243
12.2.6 Finding the Resultant Force . . . 244
12.2.7 Exercise . . . 246
12.3 Newton's Laws . . . 246
12.3.1 Newton's First Law . . . 247
12.3.2 Newton's Second Law of Motion . . . 249
12.3.3 Exercise . . . 261
12.3.4 Newton's Third Law of Motion . . . 263
12.3.5 Exercise . . . 267
12.3.6 Different types of forces . . . 268
12.3.7 Exercise . . . 275
12.3.8 Forces in equilibrium . . . 276
12.3.9 Exercise . . . 279
12.4 Forces between Masses . . . 282
12.4.1 Newton's Law of Universal Gravitation . . . 282
12.4.2 Comparative Problems . . . 284
12.4.3 Exercise . . . 286
12.5 Momentum and Impulse . . . 287
12.5.1 Vector Nature of Momentum . . . 290
12.5.2 Exercise . . . 291
12.5.3 Change in Momentum . . . 291
12.5.4 Exercise . . . 293
12.5.5 Newton's Second Law revisited . . . 293
12.5.6 Impulse . . . 294
12.5.7 Exercise . . . 296
12.5.8 Conservation of Momentum . . . 297
12.5.9 Physics in Action: Impulse . . . 300
12.5.10 Exercise . . . 301
12.6 Torque and Levers . . . 302
12.6.1 Torque . . . 302
12.6.2 Mechanical Advantage and Levers . . . 305
12.6.3 Classes of levers . . . 307
12.6.4 Exercise . . . 308
12.7 Summary . . . 309
12.8 End of Chapter exercises . . . 310

13 Geometrical Optics - Grade 11 **327**

13.1 Introduction . . . 327

13.2 Lenses . . . 327

13.2.1 Converging Lenses . . . 329

13.2.2 Diverging Lenses . . . 340

13.2.3 Summary of Image Properties . . . 343

13.3 The Human Eye . . . 344

13.3.1 Structure of the Eye . . . 345

13.3.2 Defects of Vision . . . 346

13.4 Gravitational Lenses . . . 347

13.5 Telescopes . . . 347

13.5.1 Refracting Telescopes . . . 347

13.5.2 Reflecting Telescopes . . . 348

13.5.3 Southern African Large Telescope . . . 348

13.6 Microscopes . . . 349

13.7 Summary . . . 351

13.8 Exercises . . . 352

14 Longitudinal Waves - Grade 11 **355**

14.1 Introduction . . . 355

14.2 What is a *longitudinal wave*? . . . 355

14.3 Characteristics of Longitudinal Waves . . . 356

14.3.1 Compression and Rarefaction . . . 356

14.3.2 Wavelength and Amplitude . . . 357

14.3.3 Period and Frequency . . . 357

14.3.4 Speed of a Longitudinal Wave . . . 358

14.4 Graphs of Particle Position, Displacement, Velocity and Acceleration . . . 359

14.5 Sound Waves . . . 360

14.6 Seismic Waves . . . 361

14.7 Summary - Longitudinal Waves . . . 361

14.8 Exercises - Longitudinal Waves . . . 362

15 Sound - Grade 11 **363**

15.1 Introduction . . . 363

15.2 Characteristics of a Sound Wave . . . 363

15.2.1 Pitch . . . 364

15.2.2 Loudness . . . 364

15.2.3 Tone . . . 364

15.3 Speed of Sound . . . 365

15.4 Physics of the Ear and Hearing . . . 365

15.4.1 Intensity of Sound . . . 366

15.5 Ultrasound . . . 367

15.6 SONAR . . . 368
15.6.1 Echolocation . . . 368
15.7 Summary . . . 369
15.8 Exercises . . . 369

16 The Physics of Music - Grade 11 373
16.1 Introduction . . . 373
16.2 Standing Waves in String Instruments . . . 373
16.3 Standing Waves in Wind Instruments . . . 377
16.4 Resonance . . . 382
16.5 Music and Sound Quality . . . 384
16.6 Summary - The Physics of Music . . . 385
16.7 End of Chapter Exercises . . . 386

17 Electrostatics - Grade 11 387
17.1 Introduction . . . 387
17.2 Forces between charges - Coulomb's Law . . . 387
17.3 Electric field around charges . . . 392
17.3.1 Electric field lines . . . 393
17.3.2 Positive charge acting on a test charge . . . 393
17.3.3 Combined charge distributions . . . 394
17.3.4 Parallel plates . . . 397
17.4 Electrical potential energy and potential . . . 400
17.4.1 Electrical potential . . . 400
17.4.2 Real-world application: lightning . . . 402
17.5 Capacitance and the parallel plate capacitor . . . 403
17.5.1 Capacitors and capacitance . . . 403
17.5.2 Dielectrics . . . 404
17.5.3 Physical properties of the capacitor and capacitance . . . 404
17.5.4 Electric field in a capacitor . . . 405
17.6 Capacitor as a circuit device . . . 406
17.6.1 A capacitor in a circuit . . . 406
17.6.2 Real-world applications: capacitors . . . 407
17.7 Summary . . . 407
17.8 Exercises - Electrostatics . . . 407

18 Electromagnetism - Grade 11 413
18.1 Introduction . . . 413
18.2 Magnetic field associated with a current . . . 413
18.2.1 Real-world applications . . . 418
18.3 Current induced by a changing magnetic field . . . 420
18.3.1 Real-life applications . . . 422
18.4 Transformers . . . 423

18.4.1 Real-world applications . . . 425
18.5 Motion of a charged particle in a magnetic field . . . 425
18.5.1 Real-world applications . . . 426
18.6 Summary . . . 427
18.7 End of chapter exercises . . . 427

19 Electric Circuits - Grade 11 **429**
19.1 Introduction . . . 429
19.2 Ohm's Law . . . 429
19.2.1 Definition of Ohm's Law . . . 429
19.2.2 Ohmic and non-ohmic conductors . . . 431
19.2.3 Using Ohm's Law . . . 432
19.3 Resistance . . . 433
19.3.1 Equivalent resistance . . . 433
19.3.2 Use of Ohm's Law in series and parallel Circuits . . . 438
19.3.3 Batteries and internal resistance . . . 440
19.4 Series and parallel networks of resistors . . . 442
19.5 Wheatstone bridge . . . 445
19.6 Summary . . . 447
19.7 End of chapter exercise . . . 447

20 Electronic Properties of Matter - Grade 11 **451**
20.1 Introduction . . . 451
20.2 Conduction . . . 451
20.2.1 Metals . . . 453
20.2.2 Insulator . . . 453
20.2.3 Semi-conductors . . . 454
20.3 Intrinsic Properties and Doping . . . 454
20.3.1 Surplus . . . 455
20.3.2 Deficiency . . . 455
20.4 The p-n junction . . . 457
20.4.1 Differences between p- and n-type semi-conductors . . . 457
20.4.2 The p-n Junction . . . 457
20.4.3 Unbiased . . . 457
20.4.4 Forward biased . . . 457
20.4.5 Reverse biased . . . 458
20.4.6 Real-World Applications of Semiconductors . . . 458
20.5 End of Chapter Exercises . . . 459

IV Grade 12 - Physics **461**

21 Motion in Two Dimensions - Grade 12 **463**
21.1 Introduction . . . 463

21.2 Vertical Projectile Motion . . . 463
21.2.1 Motion in a Gravitational Field . . . 463
21.2.2 Equations of Motion . . . 464
21.2.3 Graphs of Vertical Projectile Motion . . . 467
21.3 Conservation of Momentum in Two Dimensions . . . 475
21.4 Types of Collisions . . . 480
21.4.1 Elastic Collisions . . . 480
21.4.2 Inelastic Collisions . . . 485
21.5 Frames of Reference . . . 490
21.5.1 Introduction . . . 490
21.5.2 What is a *frame of reference*? . . . 491
21.5.3 Why are frames of reference important? . . . 491
21.5.4 Relative Velocity . . . 491
21.6 Summary . . . 494
21.7 End of chapter exercises . . . 495

22 Mechanical Properties of Matter - Grade 12 503
22.1 Introduction . . . 503
22.2 Deformation of materials . . . 503
22.2.1 Hooke's Law . . . 503
22.2.2 Deviation from Hooke's Law . . . 506
22.3 Elasticity, plasticity, fracture, creep . . . 508
22.3.1 Elasticity and plasticity . . . 508
22.3.2 Fracture, creep and fatigue . . . 508
22.4 Failure and strength of materials . . . 509
22.4.1 The properties of matter . . . 509
22.4.2 Structure and failure of materials . . . 509
22.4.3 Controlling the properties of materials . . . 509
22.4.4 Steps of Roman Swordsmithing . . . 510
22.5 Summary . . . 511
22.6 End of chapter exercise . . . 511

23 Work, Energy and Power - Grade 12 513
23.1 Introduction . . . 513
23.2 Work . . . 513
23.3 Energy . . . 519
23.3.1 External and Internal Forces . . . 519
23.3.2 Capacity to do Work . . . 520
23.4 Power . . . 525
23.5 Important Equations and Quantities . . . 529
23.6 End of Chapter Exercises . . . 529

24 Doppler Effect - Grade 12 **533**
24.1 Introduction . . . 533
24.2 The Doppler Effect with Sound and Ultrasound . . . 533
24.2.1 Ultrasound and the Doppler Effect . . . 537
24.3 The Doppler Effect with Light . . . 537
24.3.1 The Expanding Universe . . . 538
24.4 Summary . . . 539
24.5 End of Chapter Exercises . . . 539

25 Colour - Grade 12 **541**
25.1 Introduction . . . 541
25.2 Colour and Light . . . 541
25.2.1 Dispersion of white light . . . 544
25.3 Addition and Subtraction of Light . . . 544
25.3.1 Additive Primary Colours . . . 544
25.3.2 Subtractive Primary Colours . . . 545
25.3.3 Complementary Colours . . . 546
25.3.4 Perception of Colour . . . 546
25.3.5 Colours on a Television Screen . . . 547
25.4 Pigments and Paints . . . 548
25.4.1 Colour of opaque objects . . . 548
25.4.2 Colour of transparent objects . . . 548
25.4.3 Pigment primary colours . . . 549
25.5 End of Chapter Exercises . . . 550

26 2D and 3D Wavefronts - Grade 12 **553**
26.1 Introduction . . . 553
26.2 Wavefronts . . . 553
26.3 The Huygens Principle . . . 554
26.4 Interference . . . 556
26.5 Diffraction . . . 557
26.5.1 Diffraction through a Slit . . . 558
26.6 Shock Waves and Sonic Booms . . . 562
26.6.1 Subsonic Flight . . . 563
26.6.2 Supersonic Flight . . . 563
26.6.3 Mach Cone . . . 566
26.7 End of Chapter Exercises . . . 568

27 Wave Nature of Matter - Grade 12 **571**
27.1 Introduction . . . 571
27.2 de Broglie Wavelength . . . 571
27.3 The Electron Microscope . . . 574
27.3.1 Disadvantages of an Electron Microscope . . . 577

27.3.2 Uses of Electron Microscopes . . . 577
27.4 End of Chapter Exercises . . . 578

28 Electrodynamics - Grade 12 **579**
28.1 Introduction . . . 579
28.2 Electrical machines - generators and motors . . . 579
28.2.1 Electrical generators . . . 580
28.2.2 Electric motors . . . 582
28.2.3 Real-life applications . . . 582
28.2.4 Exercise - generators and motors . . . 584
28.3 Alternating Current . . . 585
28.3.1 Exercise - alternating current . . . 586
28.4 Capacitance and inductance . . . 586
28.4.1 Capacitance . . . 586
28.4.2 Inductance . . . 586
28.4.3 Exercise - capacitance and inductance . . . 588
28.5 Summary . . . 588
28.6 End of chapter exercise . . . 589

29 Electronics - Grade 12 **591**
29.1 Introduction . . . 591
29.2 Capacitive and Inductive Circuits . . . 591
29.3 Filters and Signal Tuning . . . 596
29.3.1 Capacitors and Inductors as Filters . . . 596
29.3.2 LRC Circuits, Resonance and Signal Tuning . . . 596
29.4 Active Circuit Elements . . . 599
29.4.1 The Diode . . . 599
29.4.2 The Light Emitting Diode (LED) . . . 601
29.4.3 Transistor . . . 603
29.4.4 The Operational Amplifier . . . 607
29.5 The Principles of Digital Electronics . . . 609
29.5.1 Logic Gates . . . 610
29.6 Using and Storing Binary Numbers . . . 616
29.6.1 Binary numbers . . . 616
29.6.2 Counting circuits . . . 617
29.6.3 Storing binary numbers . . . 619

30 EM Radiation **625**
30.1 Introduction . . . 625
30.2 Particle/wave nature of electromagnetic radiation . . . 625
30.3 The wave nature of electromagnetic radiation . . . 626
30.4 Electromagnetic spectrum . . . 626
30.5 The particle nature of electromagnetic radiation . . . 629

30.5.1 Exercise - particle nature of EM waves . . . 630
30.6 Penetrating ability of electromagnetic radiation . . . 631
30.6.1 Ultraviolet(UV) radiation and the skin . . . 631
30.6.2 Ultraviolet radiation and the eyes . . . 632
30.6.3 X-rays . . . 632
30.6.4 Gamma-rays . . . 632
30.6.5 Exercise - Penetrating ability of EM radiation . . . 633
30.7 Summary . . . 633
30.8 End of chapter exercise . . . 633

31 Optical Phenomena and Properties of Matter - Grade 12 635
31.1 Introduction . . . 635
31.2 The transmission and scattering of light . . . 635
31.2.1 Energy levels of an electron . . . 635
31.2.2 Interaction of light with metals . . . 636
31.2.3 Why is the sky blue? . . . 637
31.3 The photoelectric effect . . . 638
31.3.1 Applications of the photoelectric effect . . . 640
31.3.2 Real-life applications . . . 642
31.4 Emission and absorption spectra . . . 643
31.4.1 Emission Spectra . . . 643
31.4.2 Absorption spectra . . . 644
31.4.3 Colours and energies of electromagnetic radiation . . . 646
31.4.4 Applications of emission and absorption spectra . . . 648
31.5 Lasers . . . 650
31.5.1 How a laser works . . . 652
31.5.2 A simple laser . . . 654
31.5.3 Laser applications and safety . . . 655
31.6 Summary . . . 656
31.7 End of chapter exercise . . . 657

V Exercises 659

32 Exercises 661

VI Essays 663

Essay 1: Energy and electricity. Why the fuss? 665

33 Essay: How a cell phone works 671

34 Essay: How a Physiotherapist uses the Concept of Levers 673

35 Essay: How a Pilot Uses Vectors 675

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Chapter 11

Vectors

11.1 Introduction

This chapter focuses on vectors. We will learn what is a vector, how it differs from everyday numbers, how to add, subtract and multiply them and where they appear in Physics.

Are vectors Physics? No, vectors themselves are not Physics. Physics is just a description of the world around us. To describe something we need to use a language. The most common language used to describe Physics is Mathematics. Vectors form a very important part of the mathematical description of Physics, so much so that it is absolutely essential to master the use of vectors.

11.2 Scalars and Vectors

In Mathematics, you learned that a number is something that represents a quantity. For example if you have 5 books, 6 apples and 1 bicycle, the 5, 6, and 1 represent how many of each item you have.

These kinds of numbers are known as *scalars*.

Definition: Scalar
A scalar is a quantity that has only magnitude (size).

An extension to a scalar is a vector, which is a scalar with a direction. For example, if you travel 1 km down Main Road to school, the quantity **1 km down Main Road** is a vector. The **1 km** is the quantity (or scalar) and the **down Main Road** gives a direction.

In Physics we use the word *magnitude* to refer to the scalar part of the vector.

Definition: Vectors
A vector is a quantity that has both magnitude and direction.

A vector should tell you *how much* and *which way*.

For example, a man is driving his car east along a freeway at 100 km·hr^{-1}. What we have given here is a vector – the velocity. The car is moving at 100 km·hr^{-1}(this is the magnitude) and we know where it is going – east (this is the direction). Thus, we know the speed and direction of the car. These two quantities, a magnitude and a direction, form a vector we call velocity.

11.3 Notation

Vectors are different to scalars and therefore has its own notation.

11.3.1 Mathematical Representation

There are many ways of writing the symbol for a vector. Vectors are denoted by symbols with an arrow pointing to the right above it. For example, $\vec{a}$, $\vec{v}$ and $\vec{F}$ represent the vectors acceleration, velocity and force, meaning they have both a magnitude and a direction.

Sometimes just the magnitude of a vector is needed. In this case, the arrow is omitted. In other words, F denotes the magnitude of vector $\vec{F}$. $|\vec{F}|$ is another way of representing the magnitude of a vector.

11.3.2 Graphical Representation

Vectors are drawn as arrows. An arrow has both a magnitude (how long it is) and a direction (the direction in which it points). The starting point of a vector is known as the *tail* and the end point is known as the *head*.

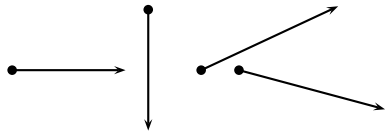


Figure 11.1: Examples of vectors

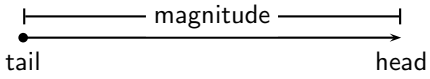


Figure 11.2: Parts of a vector

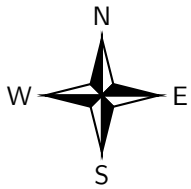
11.4 Directions

There are many acceptable methods of writing vectors. As long as the vector has a magnitude and a direction, it is most likely acceptable. These different methods come from the different methods of expressing a direction for a vector.

11.4.1 Relative Directions

The simplest method of expressing direction is relative directions: to the left, to the right, forward, backward, up and down.

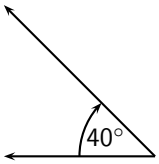
11.4.2 Compass Directions



Another common method of expressing directions is to use the points of a compass: North, South, East, and West.

If a vector does not point exactly in one of the compass directions, then we use an angle. For example, we can have a vector pointing 40° North of West. Start with the vector pointing along the West direction:

Then rotate the vector towards the north until there is a 40° angle between the vector and the West.

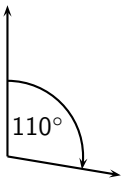


The direction of this vector can also be described as: W 40° N (West 40° North); or N 50° W (North 50° West)

11.4.3 Bearing

The final method of expressing direction is to use a *bearing*. A bearing is a direction relative to a fixed point.

Given just an angle, the convention is to define the angle with respect to the North. So, a vector with a direction of 110° has been rotated clockwise 110° relative to the North. A bearing is always written as a three digit number, for example 275° or 080° (for 80°).

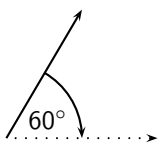


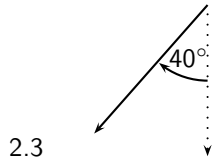
Exercise: Scalars and Vectors

1. Classify the following quantities as scalars or vectors:
   1.1 12 km
   1.2 1 m south
   1.3 2 m·s^{-1}, 45°
   1.4 075°, 2 cm
   1.5 100 km·hr^{-1}, 0°
2. Use two different notations to write down the direction of the vector in each of the following diagrams:

2.1

2.2

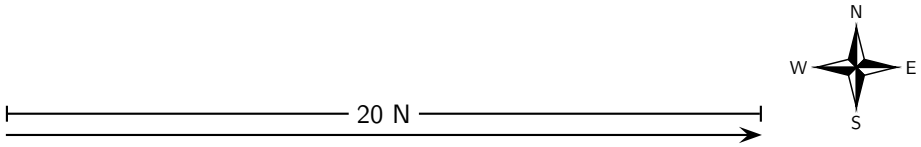




2.3

11.5 Drawing Vectors

In order to draw a vector accurately we must specify a scale and include a reference direction in the diagram. A scale allows us to translate the length of the arrow into the vector's magnitude. For instance if one chose a scale of 1 cm = 2 N (1 cm represents 2 N), a force of 20 N towards the East would be represented as an arrow 10 cm long. A reference direction may be a line representing a horizontal surface or the points of a compass.



Method: Drawing Vectors

1. Decide upon a scale and write it down.
2. Determine the length of the arrow representing the vector, by using the scale.
3. Draw the vector as an arrow. Make sure that you fill in the arrow head.
4. Fill in the magnitude of the vector.

Worked Example 49: Drawing vectors

Question: Represent the following vector quantities:

1. 6 m·s^{-1}north
2. 16 m east

Answer

Step 1 : Decide upon a scale and write it down.

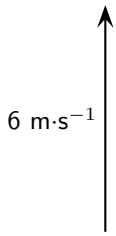
1. 1 cm = 2 m·s^{-1}
2. 1 cm = 4 m

Step 2 : Determine the length of the arrow at the specific scale.

1. If 1 cm = 2 m·s^{-1}, then 6 m·s^{-1}= 3 cm
2. If 1 cm = 4 m, then 16 m = 4 cm

Step 3 : Draw the vectors as arrows.

1. Scale used: 1 cm = 2 m·s^{-1}
   Direction = North



2. Scale used: 1 cm = 4 m
   Direction = East

Exercise: Drawing Vectors

Draw each of the following vectors to scale. Indicate the scale that you have used:

1. 12 km south
2. 1,5 m N 45° W
3. 1 m·s^{-1}, 20° East of North
4. 50 km·hr^{-1}, 085°
5. 5 mm, 225°

11.6 Mathematical Properties of Vectors

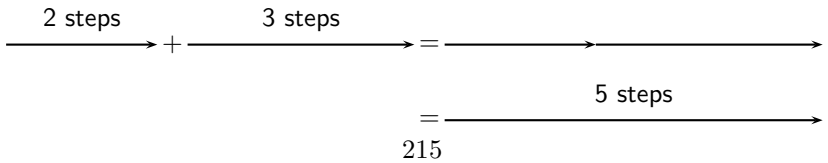
Vectors are mathematical objects and we need to understand the mathematical properties of vectors, like adding and subtracting.

For all the examples in this section, we will use displacement as our vector quantity. Displacement was discussed in Chapter 3. Displacement is defined as the distance together with direction of the straight line joining a final point to an initial point.

Remember that displacement is just one example of a vector. We could just as well have decided to use forces or velocities to illustrate the properties of vectors.

11.6.1 Adding Vectors

When vectors are added, we need to add both a magnitude **and** a direction. For example, take 2 steps in the forward direction, stop and then take another 3 steps in the forward direction. The first 2 steps is a displacement vector and the second 3 steps is also a displacement vector. If we did not stop after the first 2 steps, we would have taken 5 steps in the forward direction in total. Therefore, if we add the displacement vectors for 2 steps and 3 steps, we should get a total of 5 steps in the forward direction. Graphically, this can be seen by first following the first vector two steps forward and then following the second one three steps forward:



We add the second vector at the end of the first vector, since this is where we now are after the first vector has acted. The vector from the tail of the first vector (the starting point) to the head of the last (the end point) is then the sum of the vectors. This is the *head-to-tail* method of vector addition.

As you can convince yourself, the order in which you add vectors does not matter. In the example above, if you decided to first go 3 steps forward and then another 2 steps forward, the end result would still be 5 steps forward.

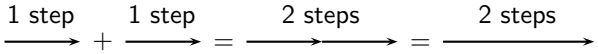
The final answer when adding vectors is called the *resultant*. The resultant displacement in this case will be 5 steps forward.

Definition: Resultant of Vectors
The resultant of a number of vectors is the single vector whose effect is the same as the individual vectors acting together.

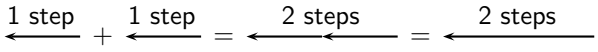
In other words, the individual vectors can be replaced by the resultant – the overall effect is the same. If vectors $\vec{a}$ and $\vec{b}$ have a resultant $\vec{R}$, this can be represented mathematically as,

$$\vec{R} \quad = \quad \vec{a} + \vec{b}.$$

Let us consider some more examples of vector addition using displacements. The arrows tell you how far to move and in what direction. Arrows to the right correspond to steps forward, while arrows to the left correspond to steps backward. Look at all of the examples below and check them.

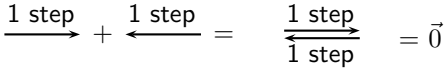


This example says 1 step forward and then another step forward is the same as an arrow twice as long – two steps forward.

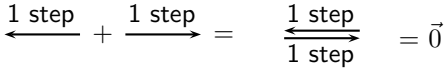


This examples says 1 step backward and then another step backward is the same as an arrow twice as long – two steps backward.

It is sometimes possible that you end up back where you started. In this case the net result of what you have done is that you have gone nowhere (your start and end points are at the same place). In this case, your resultant displacement is a vector with length zero units. We use the symbol $\vec{0}$ to denote such a vector:



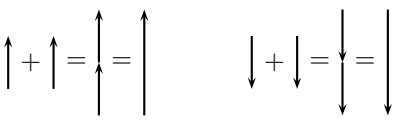
$= \vec{0}$



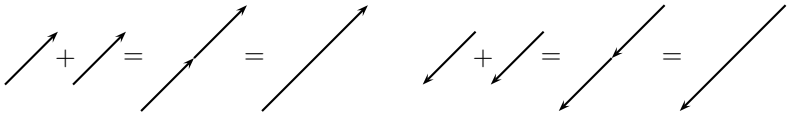
$= \vec{0}$

Check the following examples in the same way. Arrows up the page can be seen as steps left and arrows down the page as steps right.

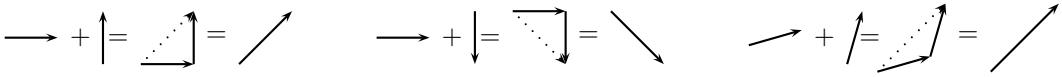
Try a couple to convince yourself!



It is important to realise that the directions are not special– 'forward and backwards' or 'left and right' are treated in the same way. The same is true of any set of parallel directions:



In the above examples the separate displacements were parallel to one another. However the same head-to-tail technique of vector addition can be applied to vectors in any direction.

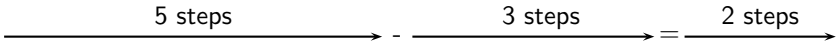


Now you have discovered one use for vectors; describing resultant displacement – how far and in what direction you have travelled after a series of movements.

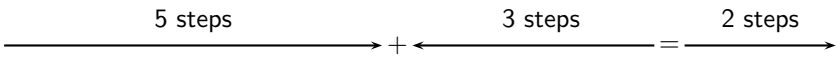
Although vector addition here has been demonstrated with displacements, all vectors behave in exactly the same way. Thus, if given a number of forces acting on a body you can use the same method to determine the resultant force acting on the body. We will return to vector addition in more detail later.

11.6.2 Subtracting Vectors

What does it mean to subtract a vector? Well this is really simple; if we have 5 apples and we subtract 3 apples, we have only 2 apples left. Now lets work in steps; if we take 5 steps forward and then subtract 3 steps forward we are left with only two steps forward:



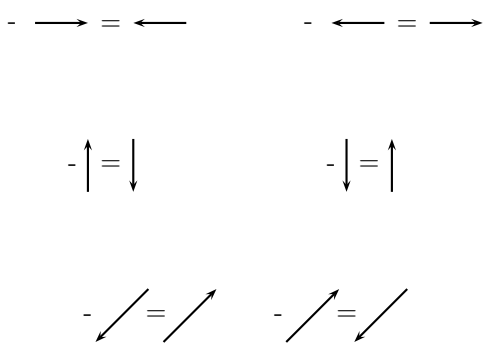
What have we done? You originally took 5 steps forward but then you took 3 steps back. That backward displacement would be represented by an arrow pointing to the left (backwards) with length 3. The net result of adding these two vectors is 2 steps forward:



Thus, subtracting a vector from another is the same as adding a vector in the opposite direction (i.e. subtracting 3 steps forwards is the same as adding 3 steps backwards).

Important: Subtracting a vector from another is the same as adding a vector in the opposite direction.

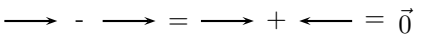
This suggests that in this problem to the right was chosen as the positive direction. Arrows to the right are positive and arrows to the left are negative. More generally, vectors in opposite directions differ in sign (i.e. if we define up as positive, then vectors acting down are negative). Thus, changing the sign of a vector simply reverses its direction:

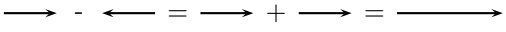


In mathematical form, subtracting $\vec{a}$ from $\vec{b}$ gives a new vector $\vec{c}$:

$$\begin{aligned} \vec{c} &= \vec{b} - \vec{a} \\ &= \vec{b} + (-\vec{a}) \end{aligned}$$

This clearly shows that subtracting vector $\vec{a}$ from $\vec{b}$ is the same as adding $(-\vec{a})$ to $\vec{b}$. Look at the following examples of vector subtraction.

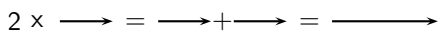




11.6.3 Scalar Multiplication

What happens when you multiply a vector by a scalar (an ordinary number)?

Going back to normal multiplication we know that 2×2 is just 2 groups of 2 added together to give 4. We can adopt a similar approach to understand how vector multiplication works.



11.7 Techniques of Vector Addition

Now that you have learned about the mathematical properties of vectors, we return to vector addition in more detail. There are a number of techniques of vector addition. These techniques fall into two main categories - graphical and algebraic techniques.

11.7.1 Graphical Techniques

Graphical techniques involve drawing accurate scale diagrams to denote individual vectors and their resultants. We next discuss the two primary graphical techniques, the head-to-tail technique and the parallelogram method.

The Head-to-Tail Method

In describing the mathematical properties of vectors we used displacements and the head-to-tail graphical method of vector addition as an illustration. The head-to-tail method of graphically adding vectors is a standard method that must be understood.

Method: Head-to-Tail Method of Vector Addition

1. Choose a scale and include a reference direction.
2. Choose any of the vectors and draw it as an arrow in the correct direction and of the correct length – remember to put an arrowhead on the end to denote its direction.
3. Take the next vector and draw it as an arrow starting from the arrowhead of the first vector in the correct direction and of the correct length.
4. Continue until you have drawn each vector – each time starting from the head of the previous vector. In this way, the vectors to be added are drawn one after the other head-to-tail.
5. The resultant is then the vector drawn from the tail of the first vector to the head of the last. Its magnitude can be determined from the length of its arrow using the scale. Its direction too can be determined from the scale diagram.

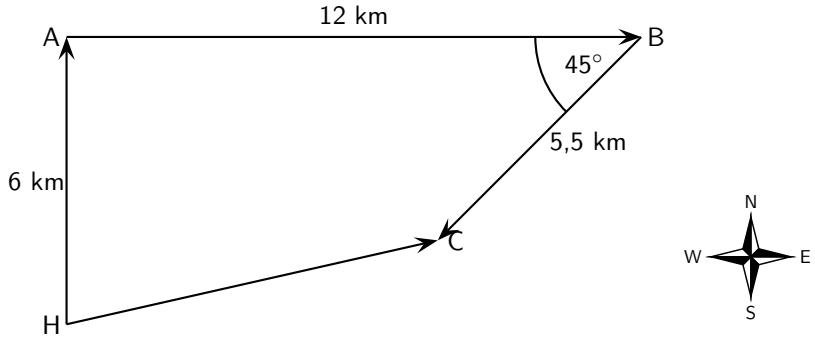
Worked Example 50: Head-to-Tail Addition I

Question: A ship leaves harbour H and sails 6 km north to port A. From here the ship travels 12 km east to port B, before sailing 5,5 km south-west to port C. Determine the ship's resultant displacement using the head-to-tail technique of vector addition.

Answer

Step 1 : Draw a rough sketch of the situation

Its easy to understand the problem if we first draw a quick sketch. The rough sketch should include all of the information given in the problem. All of the magnitudes of the displacements are shown and a compass has been included as a reference direction. In a rough sketch one is interested in the approximate shape of the vector diagram.



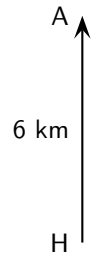
Step 2 : Choose a scale and include a reference direction.

The choice of scale depends on the actual question – you should choose a scale such that your vector diagram fits the page.

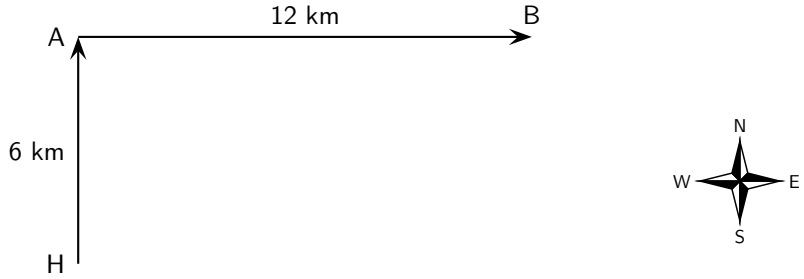
It is clear from the rough sketch that choosing a scale where 1 cm represents 2 km (scale: 1 cm = 2 km) would be a good choice in this problem. The diagram will then take up a good fraction of an A4 page. We now start the accurate construction.

Step 3 : Choose any of the vectors to be summed and draw it as an arrow in the correct direction and of the correct length – remember to put an arrowhead on the end to denote its direction.

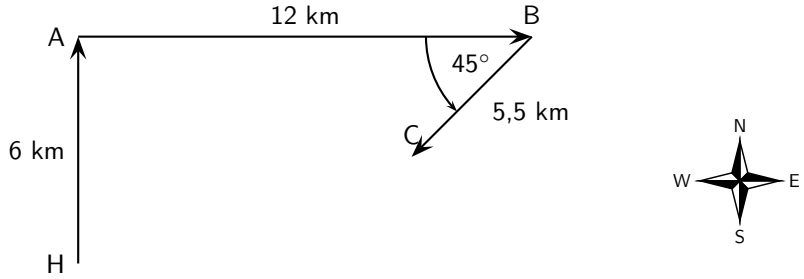
Starting at the harbour H we draw the first vector 3 cm long in the direction north.



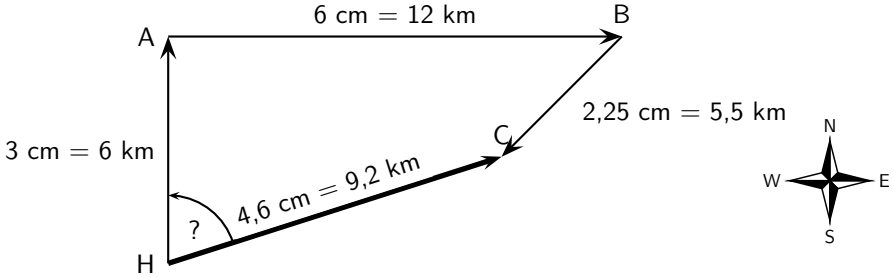
Step 4 : Take the next vector and draw it as an arrow starting from the head of the first vector in the correct direction and of the correct length.
Since the ship is now at port A we draw the second vector 6 cm long starting from point A in the direction east.



Step 5 : Take the next vector and draw it as an arrow starting from the head of the second vector in the correct direction and of the correct length.
Since the ship is now at port B we draw the third vector 2,25 cm long starting from this point in the direction south-west. A protractor is required to measure the angle of 45°.



Step 6 : The resultant is then the vector drawn from the tail of the first vector to the head of the last. Its magnitude can be determined from the length of its arrow using the scale. Its direction too can be determined from the scale diagram.
As a final step we draw the resultant displacement from the starting point (the harbour H) to the end point (port C). We use a ruler to measure the length of this arrow and a protractor to determine its direction.



Step 7 : Apply the scale conversion
We now use the scale to convert the length of the resultant in the scale diagram to the actual displacement in the problem. Since we have chosen a scale of 1 cm = 2 km in this problem the resultant has a magnitude of 9,2 km. The direction can be specified in terms of the angle measured either as 072,3° east of north or on a bearing of 072,3°.

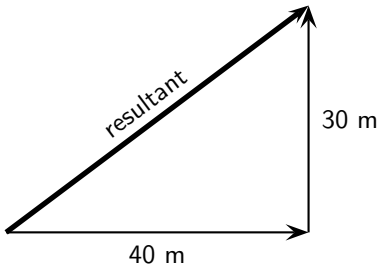
Step 8 : Quote the final answer
The resultant displacement of the ship is 9,2 km on a bearing of 072,3°.

Worked Example 51: Head-to-Tail Graphical Addition II

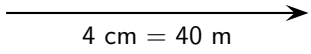
Question: A man walks 40 m East, then 30 m North.

1. What was the total distance he walked?
2. What is his resultant displacement?

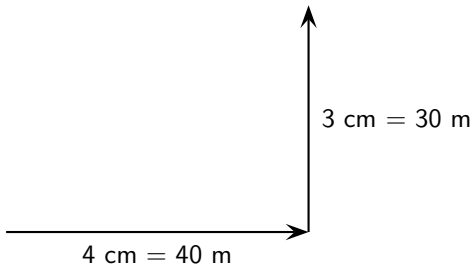
Answer
Step 1 : Draw a rough sketch



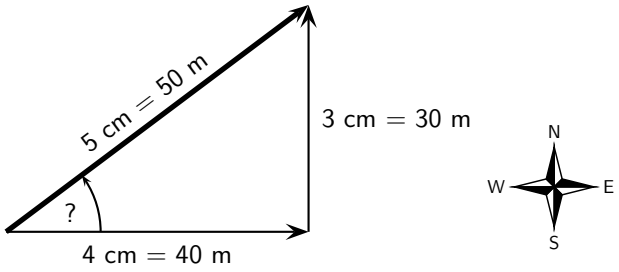
Step 2 : Determine the distance that the man traveled
In the first part of his journey he traveled 40 m and in the second part he traveled 30 m. This gives us a total distance traveled of 40 m + 30 m = 70 m.
Step 3 : Determine his resultant displacement
The man's resultant displacement is the **vector** from where he started to where he ended. It is the vector sum of his two separate displacements. We will use the head-to-tail method of accurate construction to find this vector.
Step 4 : Choose a suitable scale
A scale of 1 cm represents 10 m (1 cm = 10 m) is a good choice here. Now we can begin the process of construction.
Step 5 : Draw the first vector to scale
We draw the first displacement as an arrow 4 cm long in an eastwards direction.



Step 6 : Draw the second vector to scale
Starting from the head of the first vector we draw the second vector as an arrow 3 cm long in a northerly direction.



Step 7 : Determine the resultant vector
Now we connect the starting point to the end point and measure the length and direction of this arrow (the resultant).



Step 8 : Find the direction
To find the direction you measure the angle between the resultant and the 40 m vector. You should get about 37°.
Step 9 : Apply the scale conversion
Finally we use the scale to convert the length of the resultant in the scale diagram to the actual magnitude of the resultant displacement. According to the chosen scale 1 cm = 10 m. Therefore 5 cm represents 50 m. The resultant displacement is then 50 m 37° north of east.

The Parallelogram Method

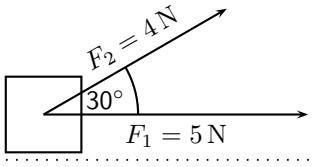
The *parallelogram method* is another graphical technique of finding the resultant of two vectors.

Method: The Parallelogram Method

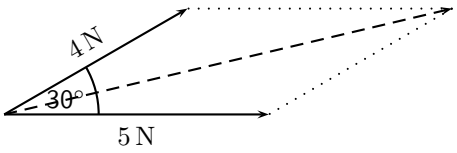
1. Choose a scale and a reference direction.
2. Choose either of the vectors to be added and draw it as an arrow of the correct length in the correct direction.
3. Draw the second vector as an arrow of the correct length in the correct direction from the tail of the first vector.
4. Complete the parallelogram formed by these two vectors.
5. The resultant is then the diagonal of the parallelogram. The magnitude can be determined from the length of its arrow using the scale. The direction too can be determined from the scale diagram.

Worked Example 52: Parallelogram Method of Vector Addition I

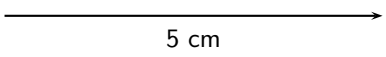
Question: A force of $F_1 = 5\,\text{N}$ is applied to a block in a horizontal direction. A second force $F_2 = 4\,\text{N}$ is applied to the object at an angle of 30° above the horizontal.



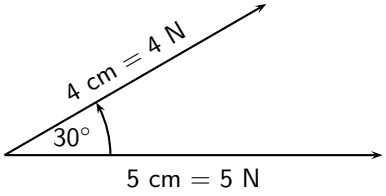
Determine the resultant force acting on the block using the parallelogram method of accurate construction.
Answer
Step 1 : Firstly make a rough sketch of the vector diagram



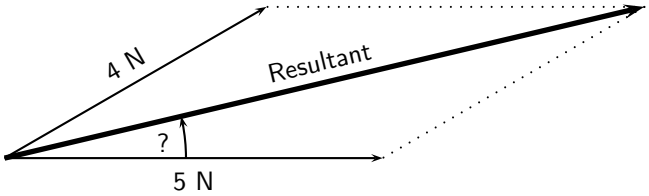
Step 2 : Choose a suitable scale
In this problem a scale of 1 cm = 1 N would be appropriate, since then the vector diagram would take up a reasonable fraction of the page. We can now begin the accurate scale diagram.
Step 3 : Draw the first scaled vector
Let us draw F_1 first. According to the scale it has length 5 cm.



Step 4 : Draw the second scaled vector
Next we draw F_2. According to the scale it has length 4 cm. We make use of a protractor to draw this vector at 30° to the horizontal.



Step 5 : Determine the resultant vector
Next we complete the parallelogram and draw the diagonal.



The resultant has a measured length of 8,7 cm.
Step 6 : Find the direction
We use a protractor to measure the angle between the horizontal and the resultant. We get 13,3°.
Step 7 : Apply the scale conversion
Finally we use the scale to convert the measured length into the actual magnitude. Since 1 cm = 1 N, 8,7 cm represents 8,7 N. Therefore the resultant force is 8,7 N at 13,3° above the horizontal.

The parallelogram method is restricted to the addition of just two vectors. However, it is arguably the most intuitive way of adding two forces acting at a point.

11.7.2 Algebraic Addition and Subtraction of Vectors

Vectors in a Straight Line

Whenever you are faced with adding vectors acting in a straight line (i.e. some directed left and some right, or some acting up and others down) you can use a very simple algebraic technique:

Method: Addition/Subtraction of Vectors in a Straight Line

1. Choose a positive direction. As an example, for situations involving displacements in the directions west and east, you might choose west as your positive direction. In that case, displacements east are negative.
2. Next simply add (or subtract) the vectors using the appropriate signs.
3. As a final step the direction of the resultant should be included in words (positive answers are in the positive direction, while negative resultants are in the negative direction).

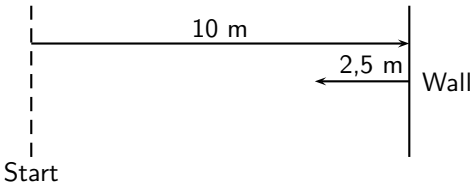
Let us consider a few examples.

Worked Example 53: Adding vectors algebraically I

Question: A tennis ball is rolled towards a wall which is 10 m away from the wall. If after striking the wall the ball rolls a further 2,5 m along the ground away from the wall, calculate algebraically the ball's resultant displacement.

Answer

Step 1 : Draw a rough sketch of the situation



Step 2 : Decide which method to use to calculate the resultant

We know that the resultant displacement of the ball ($\vec{x}_R$) is equal to the sum of the ball's separate displacements ($\vec{x}_1$ and $\vec{x}_2$):

$$\vec{x}_R = \vec{x}_1 + \vec{x}_2$$

Since the motion of the ball is in a straight line (i.e. the ball moves towards and away from the wall), we can use the method of algebraic addition just explained.

Step 3 : Choose a positive direction

Let's make towards the wall the **positive** direction. This means that away from the wall becomes the **negative** direction.

Step 4 : Now define our vectors algebraically

With right positive:

$$\begin{aligned} \vec{x}_1 &= +10{,}0\,\text{m} \\ \vec{x}_2 &= -2{,}5\,\text{m} \end{aligned}$$

Step 5 : Add the vectors

Next we simply add the two displacements to give the resultant:

$$\begin{aligned} \vec{x}_R &= (+10\,\text{m}) + (-2{,}5\,\text{m}) \\ &= (+7{,}5)\,\text{m} \end{aligned}$$

Step 6 : Quote the resultant

Finally, in this case towards the wall means positive so: $\vec{x}_R$ = 7,5 m towards the wall.

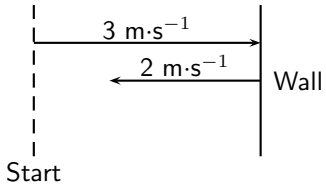
Worked Example 54: Subtracting vectors algebraically I

Question: Suppose that a tennis ball is thrown horizontally towards a wall at an initial velocity of 3 m·s^{-1}to the right. After striking the wall, the ball returns to the thrower at 2 m·s^{-1}. Determine the change in velocity of the ball.

Answer

Step 1 : Draw a sketch

A quick sketch will help us understand the problem.



Step 2 : Decide which method to use to calculate the resultant
Remember that velocity is a vector. The change in the velocity of the ball is equal to the difference between the ball's initial and final velocities:

$$\Delta\vec{v} = \vec{v}_f - \vec{v}_i$$

Since the ball moves along a straight line (i.e. left and right), we can use the algebraic technique of vector subtraction just discussed.
Step 3 : Choose a positive direction
Choose towards the wall as the **positive** direction. This means that away from the wall becomes the **negative** direction.
Step 4 : Now define our vectors algebraically

$$\begin{aligned} \vec{v}_i &= +3\,\mathrm{m}\cdot\mathrm{s}^{-1} \\ \vec{v}_f &= -2\,\mathrm{m}\cdot\mathrm{s}^{-1} \end{aligned}$$

Step 5 : Subtract the vectors
Thus, the change in velocity of the ball is:

$$\begin{aligned} \Delta\vec{v} &= (-2\,\mathrm{m}\cdot\mathrm{s}^{-1}) - (+3\,\mathrm{m}\cdot\mathrm{s}^{-1}) \\ &= (-5)\,\mathrm{m}\cdot\mathrm{s}^{-1} \end{aligned}$$

Step 6 : Quote the resultant
Remember that in this case towards the wall means positive so: $\Delta\vec{v} = 5\,\mathrm{m}\cdot\mathrm{s}^{-1}$ to the away from the wall.

?

Exercise: Resultant Vectors

1. Harold walks to school by walking 600 m Northeast and then 500 m N 40° W. Determine his resultant displacement by using accurate scale drawings.
2. A dove flies from her nest, looking for food for her chick. She flies at a velocity of 2 m·s^{-1}on a bearing of 135° and then at a velocity of 1,2 m·s^{-1}on a bearing of 230°. Calculate her resultant velocity by using accurate scale drawings.
3. A squash ball is dropped to the floor with an initial velocity of 2,5 m·s^{-1}. I rebounds (comes back up) with a velocity of 0,5 m·s^{-1}.
   3.1 What is the change in velocity of the squash ball?
   3.2 What is the resultant velocity of the squash ball?

Remember that the technique of addition and subtraction just discussed can only be applied to vectors acting along a straight line. When vectors are not in a straight line, i.e. at an angle to each other, the following method can be used:

A More General Algebraic technique

Simple geometric and trigonometric techniques can be used to find resultant vectors.

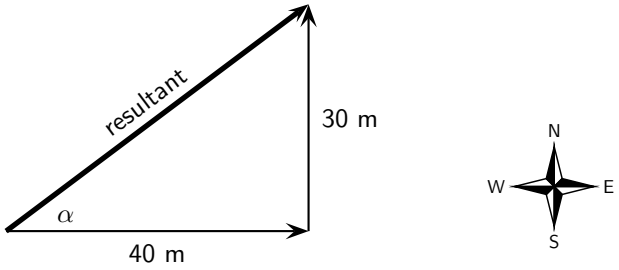
Worked Example 55: An Algebraic Solution I

Question: A man walks 40 m East, then 30 m North. Calculate the man's resultant displacement.

Answer

Step 1 : Draw a rough sketch

As before, the rough sketch looks as follows:



Step 2 : Determine the length of the resultant

Note that the triangle formed by his separate displacement vectors and his resultant displacement vector is a right-angle triangle. We can thus use the Theorem of Pythagoras to determine the length of the resultant. Let x represent the length of the resultant vector. Then:

$$\begin{aligned} x_R^2 &= (40\,\text{m})^2 + (30\,\text{m})^2 \\ x_R^2 &= 2\ 500\,\text{m}^2 \\ x_R &= 50\,\text{m} \end{aligned}$$

Step 3 : Determine the direction of the resultant

Now we have the length of the resultant displacement vector but not yet its direction. To determine its direction we calculate the angle α between the resultant displacement vector and East, by using simple trigonometry:

$$\begin{aligned} \tan\alpha &= \frac{\text{oppositeside}}{\text{adjacentside}} \\ \tan\alpha &= \frac{30}{40} \\ \alpha &= \tan^{-1}(0{,}75) \\ \alpha &= 36{,}9^\circ \end{aligned}$$

Step 4 : Quote the resultant

The resultant displacement is then 50 m at 36,9° North of East.
This is exactly the same answer we arrived at after drawing a scale diagram!

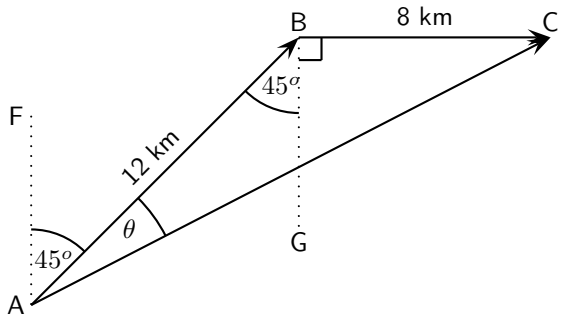
In the previous example we were able to use simple trigonometry to calculate the resultant displacement. This was possible since the directions of motion were perpendicular (north and east). Algebraic techniques, however, are not limited to cases where the vectors to be combined are along the same straight line or at right angles to one another. The following example illustrates this.

Worked Example 56: An Algebraic Solution II

Question: A man walks from point A to point B which is 12 km away on a bearing of 45°. From point B the man walks a further 8 km east to point C. Calculate the resultant displacement.

Answer

Step 1 : Draw a rough sketch of the situation



$B\hat{A}F = 45°$ since the man walks initially on a bearing of 45°. Then, $A\hat{B}G = B\hat{A}F = 45°$ (parallel lines, alternate angles). Both of these angles are included in the rough sketch.

Step 2 : Calculate the length of the resultant

The resultant is the vector AC. Since we know both the lengths of AB and BC and the included angle $A\hat{B}C$, we can use the cosine rule:

$$\begin{aligned} AC^2 &= AB^2 + BC^2 - 2 \cdot AB \cdot BC \cos(A\hat{B}C) \\ &= (12)^2 + (8)^2 - 2 \cdot (12)(8) \cos(135°) \\ &= 343{,}8 \\ AC &= 18{,}5 \text{ km} \end{aligned}$$

Step 3 : Determine the direction of the resultant

Next we use the sine rule to determine the angle θ:

$$\begin{aligned} \frac{\sin\theta}{8} &= \frac{\sin 135°}{18{,}5} \\ \sin\theta &= \frac{8 \times \sin 135°}{18{,}5} \\ \theta &= \sin^{-1}(0{,}3058) \\ \theta &= 17{,}8° \end{aligned}$$

To find $F\hat{A}C$, we add 45°. Thus, $F\hat{A}C = 62{,}8°$.

Step 4 : Quote the resultant

The resultant displacement is therefore 18,5 km on a bearing of 062,8°.

Exercise: More Resultant Vectors

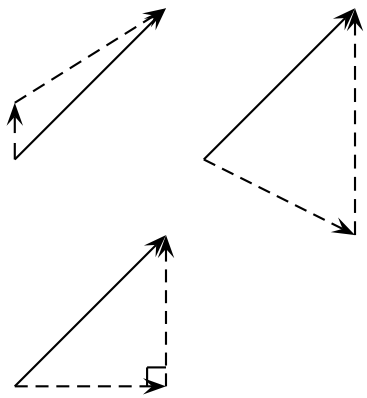
1. Hector, a long distance athlete, runs at a velocity of 3 m·s^{-1}in a northerly direction. He turns and runs at a velocity of 5 m·s^{-1}in a westerly direction. Find his resultant velocity by using appropriate calculations. Include a rough sketch of the situation in your answer.

2. Sandra walks to the shop by walking 500 m Northwest and then 400 m N 30° E. Determine her resultant displacement by doing appropriate calculations.

11.8 Components of Vectors

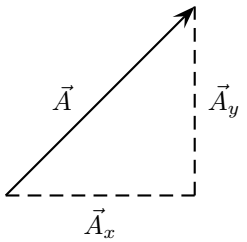
In the discussion of vector addition we saw that a number of vectors acting together can be combined to give a single vector (the resultant). In much the same way a single vector can be broken down into a number of vectors which when added give that original vector. These vectors which sum to the original are called **components** of the original vector. The process of breaking a vector into its components is called **resolving into components**.

While summing a given set of vectors gives just one answer (the resultant), a single vector can be resolved into infinitely many sets of components. In the diagrams below the same black vector is resolved into different pairs of components. These components are shown as dashed lines. When added together the dashed vectors give the original black vector (i.e. the original vector is the resultant of its components).



In practice it is most useful to resolve a vector into components which are at right angles to one another, usually horizontal and vertical.

Any vector can be resolved into a horizontal and a vertical component. If $\vec{A}$ is a vector, then the horizontal component of $\vec{A}$ is $\vec{A}_x$ and the vertical component is $\vec{A}_y$.

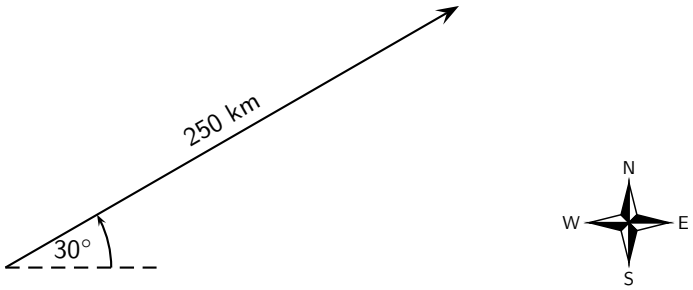


Worked Example 57: Resolving a vector into components

Question: A motorist undergoes a displacement of 250 km in a direction 30° north of east. Resolve this displacement into components in the directions north ($\vec{x}_N$) and east ($\vec{x}_E$).

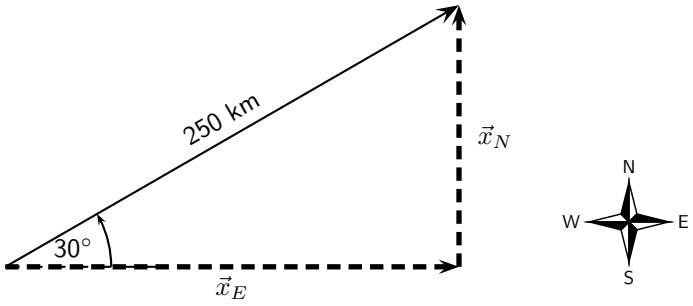
Answer

Step 1 : Draw a rough sketch of the original vector



Step 2 : Determine the vector component

Next we resolve the displacement into its components north and east. Since these directions are perpendicular to one another, the components form a right-angled triangle with the original displacement as its hypotenuse.



Notice how the two components acting together give the original vector as their resultant.

Step 3 : Determine the lengths of the component vectors

Now we can use trigonometry to calculate the magnitudes of the components of the original displacement:

$$\begin{aligned} x_N &= (250)(\sin 30^\circ) \\ &= 125 \text{ km} \end{aligned}$$

and

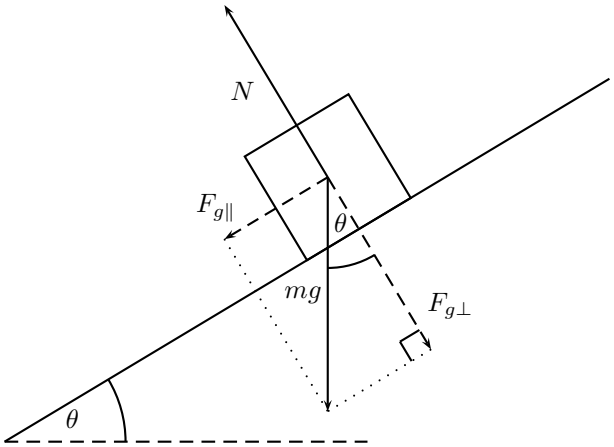
$$\begin{aligned} x_E &= (250)(\cos 30^\circ) \\ &= 216{,}5 \text{ km} \end{aligned}$$

Remember x_N and x_E are the magnitudes of the components – they are in the directions north and east respectively.

Extension: Block on an incline

As a further example of components let us consider a block of mass m placed on a frictionless surface inclined at some angle θ to the horizontal. The block will obviously slide down the incline, but what causes this motion?

The forces acting on the block are its weight mg and the normal force N exerted by the surface on the object. These two forces are shown in the diagram below.



Now the object's weight can be resolved into components parallel and perpendicular to the inclined surface. These components are shown as dashed arrows in the diagram above and are at right angles to each other. The components have been drawn acting from the same point. Applying the parallelogram method, the two components of the block's weight sum to the weight vector.

To find the components in terms of the weight we can use trigonometry:

$$\begin{aligned} F_{g\parallel} &= mg\sin\theta \\ F_{g\perp} &= mg\cos\theta \end{aligned}$$

The component of the weight perpendicular to the slope $F_{g\perp}$ exactly balances the normal force N exerted by the surface. The parallel component, however, $F_{g\parallel}$ is unbalanced and causes the block to slide down the slope.

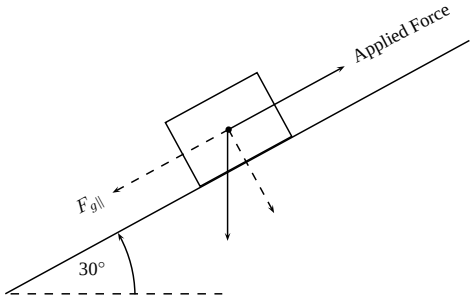
Extension: Worked example

Worked Example 58: Block on an incline plane

Question: Determine the force needed to keep a 10 kg block from sliding down a frictionless slope. The slope makes an angle of 30° with the horizontal.

Answer

Step 1 : Draw a diagram of the situation



The force that will keep the block from sliding is equal to the parallel component of the weight, but its direction is up the slope.

Step 2 : Calculate $F_{g\parallel}$

$$\begin{aligned} F_{g\parallel} &= mg\sin\theta \\ &= (10)(9{,}8)(\sin 30^\circ) \\ &= 49\text{N} \end{aligned}$$

Step 3 : Write final answer
The force is 49 N up the slope.

11.8.1 Vector addition using components

Components can also be used to find the resultant of vectors. This technique can be applied to both graphical and algebraic methods of finding the resultant. The method is simple: make a rough sketch of the problem, find the horizontal and vertical components of each vector, find the sum of all horizontal components and the sum of all the vertical components and then use them to find the resultant.

Consider the two vectors, $\vec{A}$ and $\vec{B}$, in Figure 11.3, together with their resultant, $\vec{R}$.

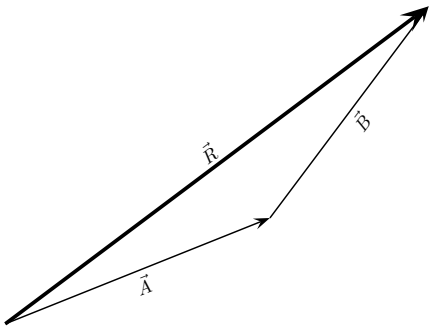


Figure 11.3: An example of two vectors being added to give a resultant

Each vector in Figure 11.3 can be broken down into a component in the x-direction and one in the y-direction. These components are two vectors which when added give you the original vector as the resultant. This is shown in Figure 11.4 where we can see that:

$$\begin{aligned} \vec{A} &= \vec{A}_x + \vec{A}_y \\ \vec{B} &= \vec{B}_x + \vec{B}_y \\ \vec{R} &= \vec{R}_x + \vec{R}_y \end{aligned}$$

$$\begin{aligned} \text{But,} \quad \vec{R}_x &= \vec{A}_x + \vec{B}_x \\ \text{and} \quad \vec{R}_y &= \vec{A}_y + \vec{B}_y \end{aligned}$$

In summary, addition of the x components of the two original vectors gives the x component of the resultant. The same applies to the y components. So if we just added all the components together we would get the same answer! This is another important property of vectors.

Worked Example 59: Adding Vectors Using Components

Question: If in Figure 11.4, $\vec{A} = 5{,}385\,\text{m}$ at an angle of 21.8° to the horizontal and $\vec{B} = 5\,\text{m}$ at an angle of 53,13° to the horizontal, find $\vec{R}$.

Answer
Step 1 : Decide how to tackle the problem

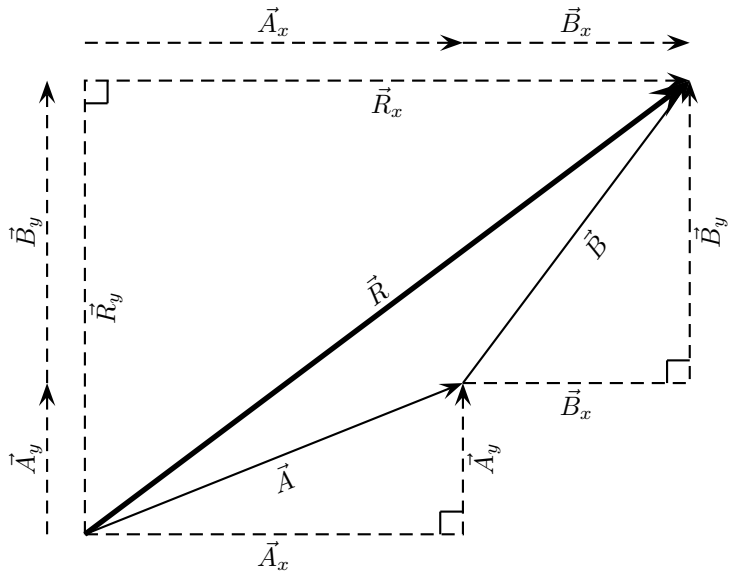


Figure 11.4: Adding vectors using components.

The first thing we must realise is that the order that we add the vectors does not matter. Therefore, we can work through the vectors to be added in any order.

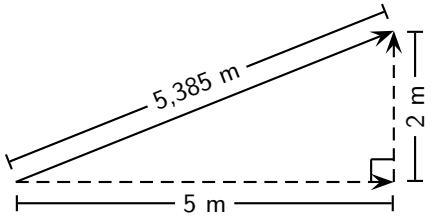
Step 2 : Resolve $\vec{A}$ into components

We find the components of $\vec{A}$ by using known trigonometric ratios. First we find the magnitude of the vertical component, A_y:

$$\begin{aligned} \sin\theta &= \frac{A_y}{A} \\ \sin 21{,}8^\circ &= \frac{A_y}{5{,}385} \\ A_y &= (5{,}385)(\sin 21{,}8^\circ) \\ &= 2\,\text{m} \end{aligned}$$

Secondly we find the magnitude of the horizontal component, A_x:

$$\begin{aligned} \cos\theta &= \frac{A_x}{A} \\ \cos 21.8^\circ &= \frac{A_x}{5{,}385} \\ A_x &= (5{,}385)(\cos 21{,}8^\circ) \\ &= 5\,\text{m} \end{aligned}$$



The components give the sides of the right angle triangle, for which the original vector is the hypotenuse.

Step 3 : Resolve $\vec{B}$ into components

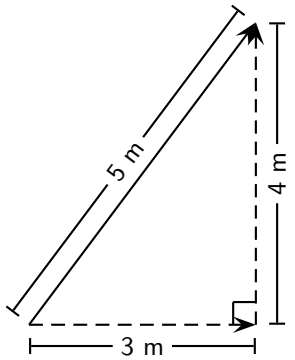
We find the components of $\vec{B}$ by using known trigonometric ratios. First we find

the magnitude of the vertical component, B_y:

$$\begin{aligned} \sin\theta &= \frac{B_y}{B} \\ \sin 53{,}13^\circ &= \frac{B_y}{5} \\ B_y &= (5)(\sin 53{,}13^\circ) \\ &= 4\,\text{m} \end{aligned}$$

Secondly we find the magnitude of the horizontal component, B_x:

$$\begin{aligned} \cos\theta &= \frac{B_x}{B} \\ \cos 21{,}8^\circ &= \frac{B_x}{5{,}385} \\ B_x &= (5{,}385)(\cos 53{,}13^\circ) \\ &= 5\,\text{m} \end{aligned}$$



Step 4 : Determine the components of the resultant vector

Now we have all the components. If we add all the horizontal components then we will have the x-component of the resultant vector, $\vec{R}_x$. Similarly, we add all the vertical components then we will have the y-component of the resultant vector, $\vec{R}_y$.

$$\begin{aligned} R_x &= A_x + B_x \\ &= 5\,\text{m} + 3\,\text{m} \\ &= 8\,\text{m} \end{aligned}$$

Therefore, $\vec{R}_x$ is 8 m to the right.

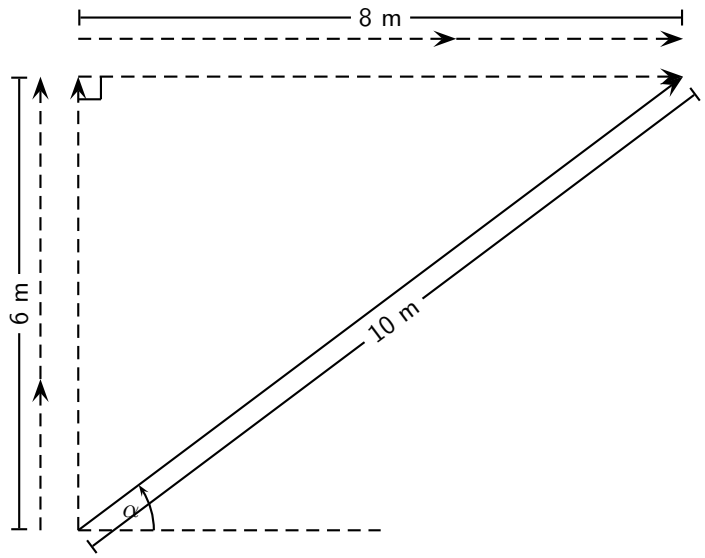
$$\begin{aligned} R_y &= A_y + B_y \\ &= 2\,\text{m} + 4\,\text{m} \\ &= 6\,\text{m} \end{aligned}$$

Therefore, $\vec{R}_y$ is 6 m up.

Step 5 : Determine the magnitude and direction of the resultant vector

Now that we have the components of the resultant, we can use the Theorem of Pythagoras to determine the magnitude of the resultant, R.

$$\begin{aligned} R^2 &= (R_x)^2 + (R_y)^2 \\ R^2 &= (6)^2 + (8)^2 \\ R^2 &= 100 \\ \therefore R &= 10\,\text{m} \end{aligned}$$



The magnitude of the resultant, R is 10 m. So all we have to do is calculate its direction. We can specify the direction as the angle the vectors makes with a known direction. To do this you only need to visualise the vector as starting at the origin of a coordinate system. We have drawn this explicitly below and the angle we will calculate is labeled α.
Using our known trigonometric ratios we can calculate the value of α;

$$\begin{aligned} \tan\alpha &= \frac{6\,\text{m}}{8\,\text{m}} \\ \alpha &= \tan^{-1}\frac{6\,\text{m}}{8\,\text{m}} \\ \alpha &= 36{,}8^{o}. \end{aligned}$$

Step 6 : Quote the final answer
$\vec{R}$ is 10 m at an angle of $36{,}8^{\circ}$ to the positive x-axis.

Exercise: Adding and Subtracting Components of Vectors

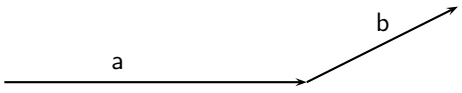
1. Harold walks to school by walking 600 m Northeast and then 500 m N 40^{o} W. Determine his resultant displacement by means of addition of components of vectors.
2. A dove flies from her nest, looking for food for her chick. She flies at a velocity of 2 m·s^{-1}on a bearing of 135^{o} and then at a velocity of 1,2 m·s^{-1}on a bearing of 230^{o}. Calculate her resultant velocity by adding the horizontal and vertical components of vectors.

Extension: Vector Multiplication
Vectors are special, they are more than just numbers. This means that multiplying vectors is not necessarily the same as just multiplying their magnitudes. There are two different types of multiplication defined for vectors. You can find the dot product of two vectors or the cross product.

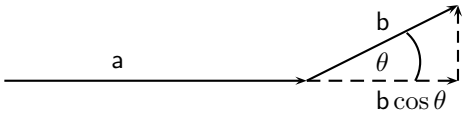
The *dot* product is most similar to regular multiplication between scalars. To take the dot product of two vectors, you just multiply their magnitudes to get out a scalar answer. The maths definition of the dot product is:

$$\vec{a} \bullet \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos\theta$$

Take two vectors $\vec{a}$ and $\vec{b}$:



You can draw in the component of $\vec{b}$ that is parallel to $\vec{a}$:



In this way we can arrive at the definition of the dot product. You find how much of $\vec{b}$ is lined up with $\vec{a}$ by finding the component of $\vec{b}$ parallel to $\vec{a}$. Then multiply the magnitude of that component, $|\vec{b}| \cos\theta$, with the magnitude of $\vec{a}$ to get a scalar.

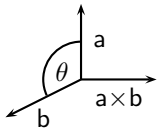
The second type of multiplication is more subtle and uses the directions of the vectors in a more complicated way to get another vector as the answer. The maths definition of the cross product is:

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta$$

This gives the magnitude of the answer, but we still need to find the direction of the resultant vector. We do this by applying the *right hand rule*.

Method: Right Hand Rule

1. Using your right hand:
2. Point your index finger in the direction of $\vec{a}$.
3. Point the middle finger in the direction of $\vec{b}$.
4. Your thumb will show the direction of $\vec{a} \times \vec{b}$.

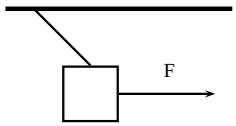


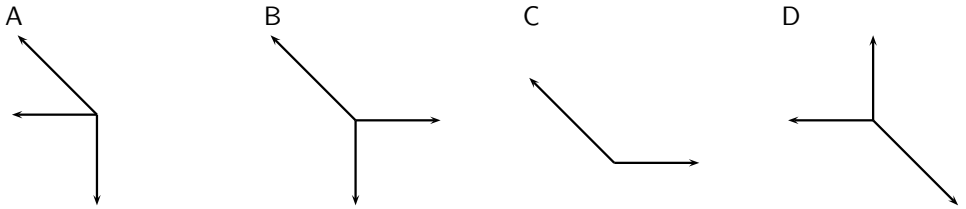
11.8.2 Summary

1. A scalar is a physical quantity with magnitude only.
2. A vector is a physical quantity with magnitude and direction.
3. Vectors are drawn as arrows where the length of the arrow indicates the magnitude and the arrowhead indicates the direction of the vector.
4. The direction of a vector can be indicated by referring to another vector or a fixed point (eg. 30° from the river bank); using a compass (eg. N 30° W); or bearing (eg. 053°).
5. Vectors can be added using the head-to-tail method, the parallelogram method or the component method.
6. The resultant of a vector is the single vector whose effect is the same as the individual vectors acting together.

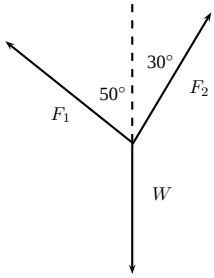
11.8.3 End of chapter exercises: Vectors

1. An object is suspended by means of a light string. The sketch shows a horizontal force F which pulls the object from the vertical position until it reaches an equilibrium position as shown. Which one of the following vector diagrams best represents all the forces acting on the object?





2. A load of weight W is suspended from two strings. F_1 and F_2 are the forces exerted by the strings on the load in the directions show in the figure above. Which one of the following equations is valid for this situation?



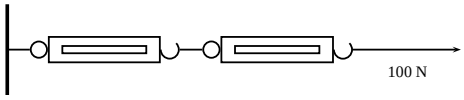
   A $W = F_1^2 + F_2^2$

   B $F_1 \sin 50° = F_2 \sin 30°$

   C $F_1 \cos 50° = F_2 \cos 30°$

   D $W = F_1 + F_2$

3. Two spring balances P and Q are connected by means of a piece of string to a wall as shown. A horizontal force of 100 N is exerted on spring balance Q. What will be the readings on spring balances P and Q?



	P	Q
A	100 N	0 N
B	25 N	75 N
C	50 N	50 N
D	100 N	100 N

4. A point is acted on by two forces in equilibrium. The forces

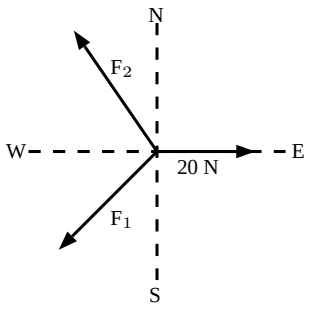
   A have equal magnitudes and directions.

   B have equal magnitudes but opposite directions.

   C act perpendicular to each other.

   D act in the same direction.

5. A point in equilibrium is acted on by three forces. Force F_1 has components 15 N due south and 13 N due west. What are the components of force F_2?



   A 13 N due north and 20 due west

   B 13 N due north and 13 N due west

   C 15 N due north and 7 N due west

   D 15 N due north and 13 N due east

6. Which of the following contains two vectors and a scalar?

   A distance, acceleration, speed

   B displacement, velocity, acceleration

   C distance, mass, speed

   D displacement, speed, velocity

7. Two vectors act on the same point. What should the angle between them be so that a maximum resultant is obtained?

A $0°$ B $90°$ C $180°$ D cannot tell

8. Two forces, 4 N and 11 N, act on a point. Which one of the following cannot be a resultant?

A 4 N B 7 N C 11 N D 15 N

11.8.4 End of chapter exercises: Vectors - Long questions

1. A helicopter flies due east with an air speed of 150 km.h^{-1}. It flies through an air current which moves at 200 km.h^{-1} north. Given this information, answer the following questions:

   1.1 In which direction does the helicopter fly?

   1.2 What is the ground speed of the helicopter?

   1.3 Calculate the ground distance covered in 40 minutes by the helicopter.

2. A plane must fly 70 km due north. A cross wind is blowing to the west at 30 km.h^{-1}. In which direction must the pilot steer if the plane goes at 200 km.h^{-1} in windless conditions?

3. A stream that is 280 m wide flows along its banks with a velocity of 1.80m.s^{-1}. A raft can travel at a speed of 2.50 m.s^{-1} across the stream. Answer the following questions:

   3.1 What is the shortest time in which the raft can cross the stream?

   3.2 How far does the raft drift downstream in that time?

   3.3 In what direction must the raft be steered against the current so that it crosses the stream perpendicular to its banks?

   3.4 How long does it take to cross the stream in question 3?

4. A helicopter is flying from place X to place Y. Y is 1000 km away in a direction $50°$ east of north and the pilot wishes to reach it in two hours. There is a wind of speed 150 km.h^{-1} blowing from the northwest. Find, by accurate construction and measurement (with a scale of $1 \text{ cm} = 50 \text{ km.h}^{-1}$), the

   4.1 the direction in which the helicopter must fly, and

   4.2 the magnitude of the velocity required for it to reach its destination on time.

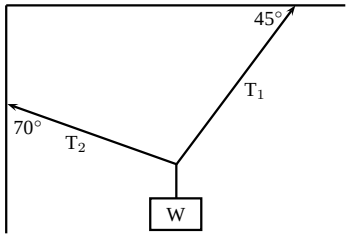
5. An aeroplane is flying towards a destination 300 km due south from its present position. There is a wind blowing from the north east at 120 km.h^{-1}. The aeroplane needs to reach its destination in 30 minutes. Find, by accurate construction and measurement (with a scale of $1 \text{ cm} = 30 \text{ km.s}^{-1}$), or otherwise, the

   5.1 the direction in which the aeroplane must fly and

   5.2 the speed which the aeroplane must maintain in order to reach the destination on time.

   5.3 Confirm your answers in the previous 2 subquestions with calculations.

6. An object of weight W is supported by two cables attached to the ceiling and wall as shown. The tensions in the two cables are T_1 and T_2 respectively. Tension $T_1 = 1200$ N. Determine the tension T_2 and weight W of the object by accurate construction and measurement or by calculation.



7. In a map-work exercise, hikers are required to walk from a tree marked A on the map to another tree marked B which lies 2,0 km due East of A. The hikers then walk in a straight line to a waterfall in position C which has components measured from B of 1,0 km E and 4,0 km N.

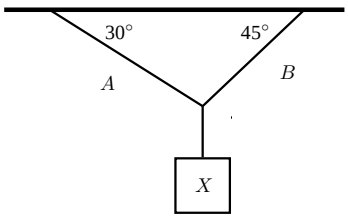
   7.1 Distinguish between quantities that are described as being *vector* and *scalar*.

   7.2 Draw a labelled displacement-vector diagram (not necessarily to scale) of the hikers' complete journey.

   7.3 What is the total distance walked by the hikers from their starting point at A to the waterfall C?

   7.4 What are the magnitude and bearing, to the nearest degree, of the displacement of the hikers from their starting point to the waterfall?

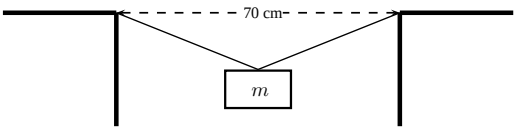
8. An object X is supported by two strings, A and B, attached to the ceiling as shown in the sketch. Each of these strings can withstand a maximum force of 700 N. The weight of X is increased gradually.



   8.1 Draw a rough sketch of the triangle of forces, and use it to explain which string will break first.

   8.2 Determine the maximum weight of X which can be supported.

9. A rope is tied at two points which are 70 cm apart from each other, on the same horizontal line. The total length of rope is 1 m, and the maximum tension it can withstand in any part is 1000 N. Find the largest mass (m), in kg, that can be carried at the midpoint of the rope, without breaking the rope. Include a vector diagram in your answer.



Appendix A

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